2014 John O'Bryan Mathematical Competition
5-person Team Test

Abbreviated Instructions: Answer each of the following questions **using separate sheet(s) of paper for each numbered problem**. Place your team letter in the upper right corner of each page that will be turned in (failure to do this will result in no score). Place problem numbers in the upper left corner. Problems are equally weighted; **teams must show complete solutions (not just answers) to receive credit**. More specific instructions are read verbally at the beginning of the test.

1. Consider an n x n x n cube, where n is a positive integer. Paint the exterior of the cube and cut the cube into n^3 unit cubes.

   a. For n=3, find the numbers of cubes having 1,2, and 3 painted sides. Additionally, how many total faces would have no paint?

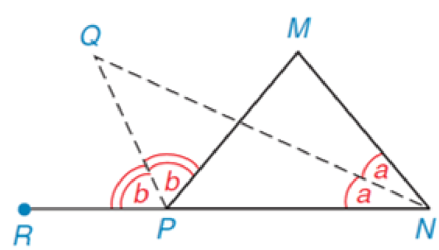
   The 3 x 3 x 3 cube would be cut into 27 unit cubes. A small amount of logic can assist in easily determining:

   - There are EIGHT cubes that have 3 painted sides (and this will be true regardless of n).
   - There is ONE cube that has no paint (the one in the very center of the cube.
   - There are TWELVE cubes that have 2 painted sides (4 from the top layer, 4 from the middle layer, and 4 from the bottom layer.
   - By subtraction, there are SIX cubes that have only one painted side (or you can get this by recognizing that the middle cube on each of the six sides of the larger cube will have one painted side.

   b. For general n, find the number of cubes having at least one painted side as a function of n.

   The number of unexposed cubes for the above case with $n = 3$ was 1 (the one in the very center). If $n = 4$, it isn't too difficult to see that the central core will consist of a 2 x 2 x 2 cube so that there would be 8 cubes with no paint. Likewise, if $n = 5$, there will be 3 x 3 x 3 = 27 central cubes. In general, there will be $(n-2)^3$ central cubes having no paint. Hence the number of cubes having at least one painted side is given by: $n^3 - (n-2)^3$. Alternatively this may be rewritten as $6n^2 - 12n + 8$.

2. In the diagram at the right, $\overrightarrow{NQ}$ bisects $\angle MNP$ and $\overrightarrow{PQ}$ bisects $\angle MPR$. Supposing that the measure of $\angle PQN$ is 42 degrees, find the measure of $\angle PMN$.



   Label $\angle PMN$ = c and this problem may be reduced to some algebra. From ΔPMN we have that

   $2a + (180 - 2b) + c = 180$ or $c = 2(b - a)$

   From ΔQPN we get:

   $42 + (180 - b) + a = 180$ or $42 = b - a$

   Substituting this into the earlier equation yields $c = 84$ degrees.

3. Fourteen poker chips – two for each of the seven colors in the rainbow – are stacked randomly in a pile.

    a. What is the probability that the blue chips are not adjacent?

    There are two "blue" chips and twelve "non-blue" chips. The number of ways to choose the locations of the blue chips in general is C(14,2). Of these, C(13,1) have the blue chips being adjacent. Hence the probability is [C(14,2) – C(13,1)] / C(14,2) = 6/7 (or 0.8571).

    b. What is the probability that the blue chips are not adjacent and the red chips are adjacent?

    Following a similar line of reasoning, there are C(14,2)*C(12,2) ways to choose the locations of the blue chips and the red chips. But if the red chips are to be adjacent, we can treat them as a single chip. Thus there are C(13,2)*C(11,1) ways to arrange the two blue chips and the red stack. Of those, C(12,1)*C(11,1) also have the blue chips adjacent. Therefore the probability is [C(13,2)*C(11,1) – C(12,1)*C(11,1)] / [C(14,2)*C(12,2)] = 11/91 = 0.1209.

    c. What is the probability that the red chips are adjacent and **both** blue chips are below the red chips in the stack?

    Again the denominator is C(14,2)*C(12,2) to choose the locations of the red and blue chips. The numerator is more difficult. There are C(13,1) possible locations for the stack of adjacent red chips. But each of these leads to a different number of possibilities for the blue chips, so we must consider them separately. If the red chips are on top, the number of possibilities for the blue chips is C(12,2). If there is one chip above the red chips, the number of possibilities for the blue chips is C(11,2). If there are two chips above the red chips, this becomes C(10,2) and so the pattern continues down to C(2,2). Hence the numerator will be $\sum_{k=2}^{12} C(k,2)$ = 286. Upon computation, the probability of this event is 0.04762 or 1/21.

NOTE: This is one valid solution to this problem; there are others.

4. You may work part (a) for five points or part (b) for ten points. **Only hand in work for one of these two problems** and clearly designate which problem you are handing in.

    a. A jar filled with honey weighs 500 grams. The same jar filled with water weighs 350 grams. If honey weighs twice as much as water (is twice as dense), find the weight of the jar.

    This is an algebra problem. Let x be the weight of the jar and y be the weight of the water. Then we have $x + y = 350$ and $x + 2y = 500$. Solving this system of equations results in $y = 150$ and hence $x = 200$. The jar weighs 200 grams.

    b. Show that $ac + bd = 0$ for the following system of equations:

$$\begin{cases} a+b+c+d=\sqrt{2} \\ a^2+b^2+c^2+d^2=1 \\ a-b+c-d=0 \end{cases}$$

    As in part (a) the goal here is to manipulate a system of equations; however this system is a bit more complicated. For convenience, label the equations (1), (2), and (3), respectively.

    By addition of equations (1) and (3) we obtain that $a + c = \frac{1}{2}\sqrt{2}$. (4)

    By subtraction of equations (1) and (3) we obtain that $b + d = \frac{1}{2}\sqrt{2}$ (5)

By squaring (4) we get $a^2 + 2ac + c^2 = \frac{1}{2}$ (6)

By squaring (5) we get $b^2 + 2bd + d^2 = \frac{1}{2}$ (7)

Summing (6) and (7) yields $a^2 + 2ac + c^2 + b^2 + 2bd + d^2 = 1$ (8)

We can now substitute (2) into equation (8) to get $2ac + 2bd + 1 = 1$ (9)

But now equation (9) immediately reduces to our goal: Thus $ac + bd = 0$.

5. For this problem to achieve full credit you must do **both** parts:

a. Assuming that log is the base 10 logarithm, solve for x if $x^{\log x} = \frac{x^3}{100}$.

First take the log of both sides of this equation, resulting in:

$$\log x^{\log x} = \log\left(\frac{x^3}{100}\right)$$

Simplification using various rules of logs and noting that log(100) = 2 yields:

$$(\log x)^2 - 3\log(x) + 2 = 0$$

This equation can be factored into:

$$(\log x - 2)(\log x - 1) = 0$$

So either $\log(x) = 1$ or $\log(x) = 2$, implying that $x = 10$ or $x = 100$. Checking both of these in the original equation shows that both are valid solutions.

b. Find the value of $y = \sqrt{12 + \sqrt{12 + \sqrt{12 + \sqrt{12 + \ldots}}}}$

Substitute y into itself to obtain:

$$y = \sqrt{12 + y}$$

Squaring both sides yields

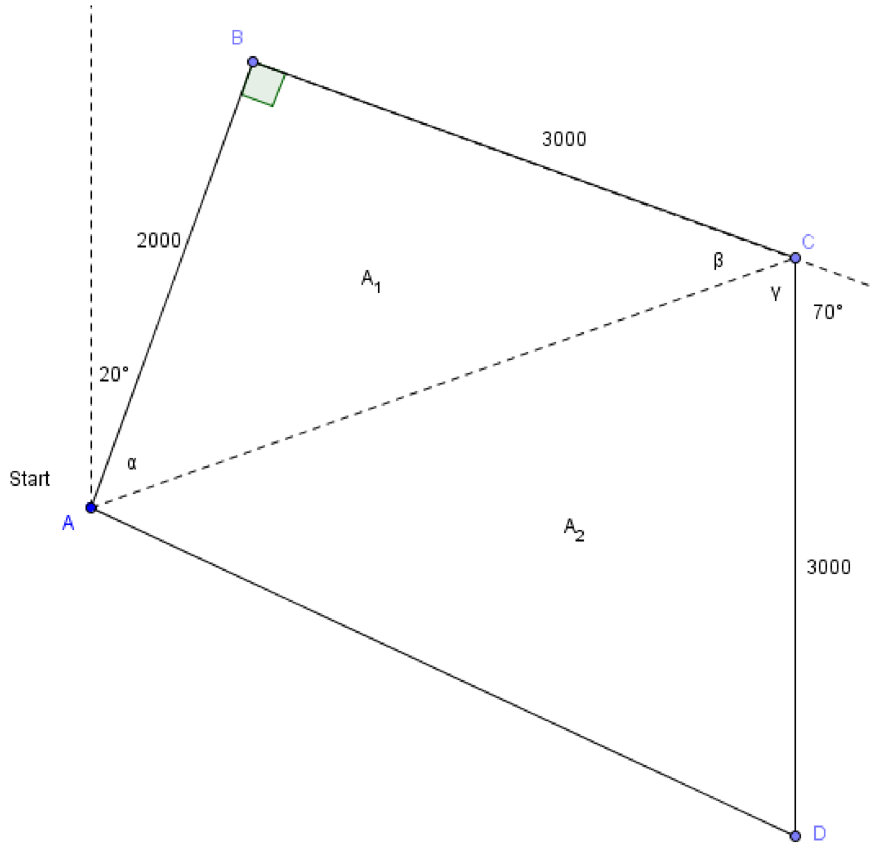
$$y^2 - y - 12 = 0$$

This factors into $(y - 4)$ and $(y + 3)$. Hence either $y = 4$ or $y = -3$ But by definition a square root may not be negative; hence $y = 4$ is the only solution.

Note: It was possible to figure out the answer by repeated operations using a calculator. However, providing the answer via a calculator (on any part of this event) does not constitute an acceptable solution.

6. A surveyor forms the boundary of a property in the following way. He walks 2000 feet in the direction 20 degrees to the east of north. He then turns 90 degrees clockwise and walks 3000 feet. Next he turns 70 degrees clockwise and walks 3000 feet. Lastly, he returns directly to his starting point. Find the area of the enclosed plot of land.

First, a drawing will help:



Several solutions are possible. Following is one possible solution.

By the Pythagorean Theorem applied to $\triangle ABC$, $AC = 1000\sqrt{13}$.

Also, $\alpha = \tan^{-1}\left(\frac{3}{2}\right) \approx 56.31°$ and $\beta = \tan^{-1}\left(\frac{2}{3}\right) \approx 33.69°$. So, $\gamma = 180 - 70 - \beta \approx 76.31°$.

Now, by the Law of Cosines, $AD = \sqrt{3000^2 + \left(1000\sqrt{13}\right)^2 - 2 \cdot 3000 \cdot 1000\sqrt{13}\cos\left(76.31°\right)} \approx 4108.54$.

The total area $A = A_1 + A_2$.

$$A_1 = \frac{1}{2} \cdot 2000 \cdot 3000 = 3{,}000{,}000 \text{ ft}^2.$$

The semi-perimeter of $\triangle ACD$ is $s = \frac{1}{2}(3000 + 1000\sqrt{13} + 4108.54)$.

So, by Heron's Formula, $A_2 = \sqrt{s(s-3000)(s-1000\sqrt{13})(s-4108.54)} \approx 5{,}254{,}680$ ft^2.

So, $A \approx 8{,}254{,}680$ ft^2.