2011 John O’Bryan Mathematical Competition
Junior-Senior 5-person Team Test

Abbreviated Instructions: Answer each of the following questions using separate sheet(s) of paper for each numbered problem. Place your team letter in the upper right corner of each page that will be turned in (failure to do this will result in no score). Problems are equally weighted; teams must show complete solutions (not just answers) to receive credit. More specific instructions are read verbally at the beginning of the test.

1. Two identical squares overlap with the corner of one square at the center of the other.
   
   a. Find the proportion of the area of each square that is in the overlapped section.
   b. Prove that the proportion of area in part (a) is independent of the relative positions of the two squares.
   c. If we replace the squares by identical equilateral triangles, with the corner of one at the center of the other, is the overlap still independent of relative positioning? Explain!

2. Two railroad cars (labeled A and B) are in a siding as shown. An engine (labeled E) is on the main track as shown. Either end of the engine can hitch to either end of each car to pull (or push) it. Of course the cars may be hitched together as well. While each car will fit alone onto the spur, there is not room for two cars, nor for the engine, on the spur. Can the engine switch the position of the two cars while also ending at the same location on the main track where it started?

3. The Fibonacci numbers begin 1, 1, 2, 3, 5, 8, ..., where each element is the sum of the previous two elements.
   
   a. Which Fibonacci numbers are even?
   b. Prove your result from part (a).
   c. Which Fibonacci numbers have 3 as a factor?
   d. Prove your result from part (c).

4. As a decimal, the fraction \( \frac{1}{97} \) has a repetend (the part that repeats) that begins after the decimal point and is 96 decimals long. If the last three digits of the repetend are A67, compute A.

5. Let \( a \) and \( b \) be two randomly chosen positive integers (not necessarily distinct) chosen such \( a \leq 100 \) and \( b \leq 100 \). What is the probability that the units digit of \( 3^a + 7^b \) is 6?

6. What is the largest positive integer \( n \) such that \( 2011! \) is divisible by \( 15^n \)? Note: \( 2011! \) is the product of every positive integer less than or equal to 2011. For example, \( 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 \).

7. If \( \sin x + \cos x = \frac{1}{2} \), find the value of \((\sin x)^4 + (\cos x)^4\).
By examining the pattern we observe that the product of the adjacent terms follows the expression $a^2 + 2ab + b^2 = (a + b)^2$. Substituting this expression into the previous expression gives $a^2 + 2ab + b^2 = (a + b)^2$. We thus have that $a^2 + 2ab + b^2 = a^2 + b^2$. We observe that $a^2 + b^2$ is always greater than $a^2 + ab + b^2$. But $a^2 + b^2 = (a + b)^2$. Therefore, $a^2 + 2ab + b^2 = (a + b)^2$ is true. We have established that $a^2 + 2ab + b^2 = (a + b)^2$.

Keep track of the number (starting with the first) as shown.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 609, 985, 1690, 2677, 4367, 7044, 11411, 18458, 29879, 48337, 78216, 126553, 204770, 331323, 536093, 867416, 1403509, 2270925, 3674434, 5945359, 9619793, 15565152, 25184945, 40750097, 65935042, 106685139, 172620181, 283305320, 456025401, 740330721, 1236356122, 2007276773, 3245603935, 5389399108, 8634998843, 14084397951, 22724396804, 37808794755, 62153180659, 100000000000
We know that $107 = 4 \times 26 + 1$.

Compute $\frac{67}{26}$ as a fraction (the part that repeats) then begin after the decimal point and add 26 decimals. Note that the last two digits of the repeating are 67.

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If we can throw possibilities only the face $g$ is 2. So the probability is $\frac{2}{6}$.

Let $a$ and $b$ be two - not necessarily distinct - numbers from 1 to 100 (inclusive). What is the probability that the units digit of $a + b$ is
So there are 40+30+16=86. We will find out how many of these are 5. There will be even more.

Since 15 = 3 x 5, we need only find how many times each of these numbers divides into 2011. Since 5 is the larger of the two factors, we will find out how many 5s there are in 2011. Since 5 is the larger of the two factors, we will find out how many 5s there are in 2011. Since 5 is the larger of the two factors, we will find out how many 5s there are in 2011.
\[
\frac{2\pi}{3} = x \cos \frac{\pi}{3}
\]

So, \(\sin x \cos \frac{\pi}{3} \geq 0\)

\[
\frac{8}{\varepsilon} = \varepsilon \
\]

So \(\sin x \cos x \). \(\varepsilon \) is small.

\[
\varepsilon (x \cos x) + \varepsilon (x \sin x) = x \cos x + x \sin x \
\]

\(\varepsilon\) is small.