2017 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. If $\frac{2+\sqrt{3}}{4-\sqrt{3}} = \frac{a+b\sqrt{3}}{c}$, where a, b, and c are integers with no common factors and $c > 0$, find the value of a.

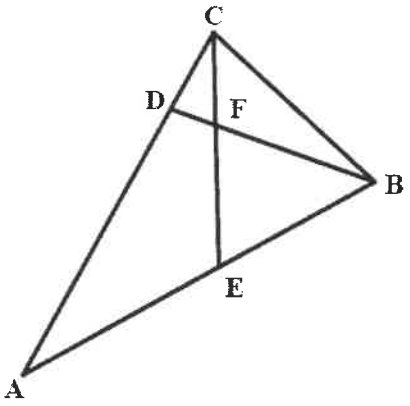
2. Given that $i - 2i^2 + 3i^3 - 4i^4 = a + bi$, where $i = \sqrt{-1}$ and a and b are real numbers, find the value of $a^2 - b^2$.

3. The first term of an arithmetic sequence is 32 and the last term is 74. There are 38 terms in the sequence. Find the sum of the 38 terms in this sequence.

4. Find the exact area of an equilateral triangle inscribed in the circle with equation $x^2 + y^2 = 12$.

5. Find the largest number of pigeonholes which 225 pigeons can occupy, given that there must be at least one pigeon in each hole and that no two holes can contain the same number of pigeons.

6. ΔABC is isosceles with vertex angle A having a measure of 36°. $\overline{CE}$ bisects $\angle ACB$ and $\overline{BD}$ is one of the trisectors of $\angle ABC$ so that the measure of $\angle CBD$ is less than the measure of $\angle DBA$. Find the degree measure of $\angle CFB$.



7. Find the ordered pair of positive integers (x, y) for which $x^2 - y^2 = 19$.

8. The graph of $4x^2 - y^2 - 8x + 2y = 1$ is rotated 90° clockwise about the origin. Find the coordinates of the center of the new rotated conic. Write your answer as the ordered pair (x, y).

9. Find the value of k for which $x + 4$ is a factor of $2x^2 + kx - 8$.

10. A bag contains red and green marbles. When two marbles are drawn, the probability that they are both red is equal to the probability that they are both green. The probability that one of each color is drawn is $\frac{4}{7}$. Find the total number of marbles in the bag.

11. Find the exact sum of an infinite geometric series whose first term is 24 and whose common ratio is $\frac{1}{5}$.

12. $\overline{AB}$ is a chord in circle O such that the degree measure of minor arc $\overset{\frown}{AB}$ is $\frac{1}{8}$ the degree measure of major arc $\overset{\frown}{AB}$. Find the degree measure of $\angle OAB$.

13. Evaluate the expression $100^2 - 99^2 + 98^2 - 97^2 + \cdots + 4^2 - 3^2 + 2^2 - 1^2$.

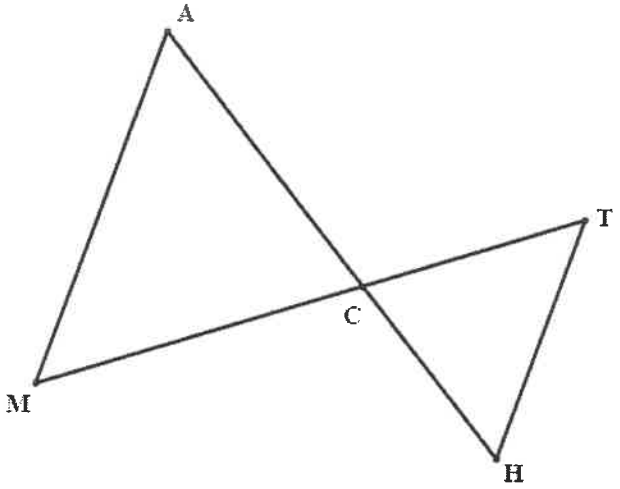
14. A point x_0 is called a fixed point of a function f if $f(x_0) = x_0$. Find the sum of all values of x which are fixed points of $f(x) = x^2 - 5x - 7$.

15. Three distinct integers are chosen from the first nine positive integers. Find the probability that 3 is the smallest of the integers chosen. Write your answer as a common fraction reduced to lowest terms.

16. Find the exact value of the tangent of the largest angle of a triangle if the lengths of the sides are 3, 7, and 8.

17. Determine the exact sum of all values of θ (in radian measure), $0 \le \theta < 2\pi$, for which $\sin^2\theta = \frac{4}{5}$.

18. In the diagram, $\overline{MA} \parallel \overline{TH}$, $MA = 4x$, $AC = 5x + 2$, $MC = 3x + 4$, $CT = x + 3$, and $TH = x + 2$. Find the length of CH.



19. Determine the number of distinct primes that are factors of 25!.

20. The composite function $f(g(x))$ is undefined for $x = \frac{k}{18}$. If $f(x) = \frac{x-1}{2x+3}$ and $g(x) = 3x - 1$, find the value of k.

Name: _____ANSWERS_____ Team Code: _____

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1. 11
2. 0
3. 2014
4. $9\sqrt{3}$
5. 20
6. 120
7. $(10,9)$
8. $(1,-1)$
9. 6
10. 8
11. 30
12. 70
13. 5050
14. 6
15. $\frac{5}{28}$
16. $-4\sqrt{3}$
17. 4π
18. 6
19. 9
20. -3