

2017 John O'Bryan Mathematics Competition 5-Person Team Test

Abbreviated Instructions: Answer each of the following questions **using separate sheet(s) of paper for each numbered problem**. Place your team code in the upper right corner of each page that will be turned in (failure to do this will result in no score). Place problem numbers in the upper left corner. Problems are equally weighted; **teams must show complete solutions (not just answers) to receive credit**. More specific instructions are read verbally at the beginning of the test.

1. Complete each of the following (5 points each part):

a. Assume $x \neq 1$. Find all solutions to the equation: $(x^2 + x + 1)(x^6 + x^3 + 1) = \frac{10}{x - 1}$

b. Let $f(x) = x^2 + ax - 9$ and $g(x) = x^2 + bx - c$ where a, b , and c are constants.

Suppose that $f(x)$ has roots r and s while $g(x)$ has roots $-r$ and $-s$. Find the roots of $f(x) + g(x)$

2. Let $T(x)$ be a non-zero polynomial. Under what conditions on T do we have each of the following properties?

a. $T(T(x)) = 1$

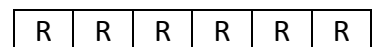
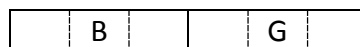
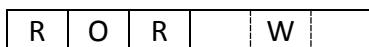
b. $T(T(x)) = x$

c. $T(T(x)) = T(x)$

d. $T(T(x)) = (T(x))^2$

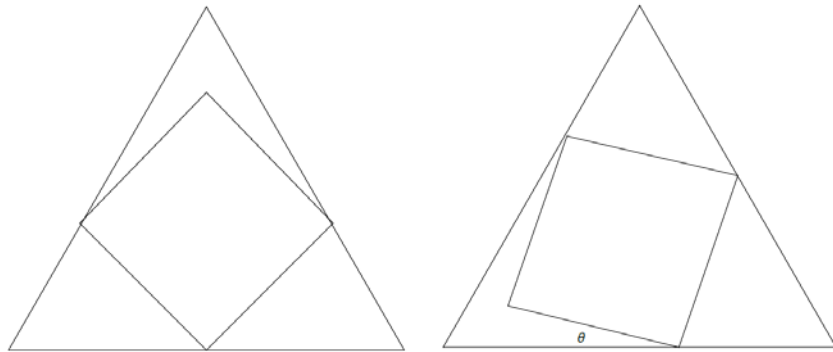
3. You are given two sizes of ceramic tiles. There are 1×1 tiles, colored either white or red. And there are 1×3 tiles, colored blue, green, or orange. For each of the 5 different tiles, assume that you have as many as you need.

You are able to make patterns by stringing tiles together. For example, you can make a 1×6 pattern of red, orange, red, and white. Of course you could also make a 1×6 pattern using blue followed by green. You are also allowed to be boring and tile the 1×6 using all red tiles, if you wish. These three examples are illustrated below.

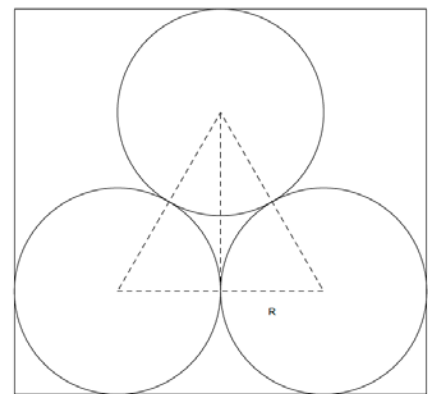


How many different $1 \times n$ patterns can be made from these tiles, for $n = 1, 2, 3, 4, 5$, and 6 ?

4. The numbers from 1 to 1000 are written, in order, in a large circle. Starting at the number 1, every r^{th} number (1, $1+r$, $1+2r$, etc.) is crossed out. This is continued until a number is reached that has already been crossed out.
- If $r = 15$, what is the total number of cross-outs?
 - In general, what is the total number of cross-outs?
5. An equilateral triangle with sides of length 2 will have a square placed inside of it.
- If one side of the square sits on one side of the triangle, find the area of the largest such square.
 - If the square is oriented such that one diagonal of the square is collinear with one of the vertices of the triangle (as shown below, left), find the area of the largest such square.
 - Finally, if the triangle is as shown below (right), with $0 \leq \theta \leq 30$ degrees, find a formula for the area of the largest such square at angle θ .



6. In the figure shown, the circles are tangent to one another and to the sides of the rectangle. Each of the circles has radius R .
- What is the area of the entire rectangle?
 - What is the area of the region trapped between the three circles?



Solutions to team exam :

Note to coaches : I have inserted some comments after a few of the problems. You may find these useful in preparing your teams in the future. I noticed that a lot of teams chose methods that were more difficult than needed. It might be useful for teams to spend time looking at each problem and guessing at possible best strategies as a team, before dividing up problems to be solved by individuals.

1 a. Assuming $x \neq 1$, find the solutions to $(x^2 + x + 1)(x^6 + x^3 + 1) = \frac{10}{x - 1}$

Solution : Multiply $(x-1)$ times $(x^2 + x + 1)$ to get $x^3 - 1$. Now multiply this times $(x^6 + x^3 + 1)$ to get $x^9 - 1 = 10$ and so $x = \sqrt[9]{11}$

Note: Many teams did this by multiplying the left side out, getting a large mess, and then simplifying down. Notice how recognizing the difference of cubes makes this problem a lot easier, saving time and eliminating many possible errors.

b. If $f(x) = x^2 + bx - 9$ and $g(x) = x^2 + dx - e$, and if f has two roots r and s , while g has roots $-r$ and $-s$, find the roots of $f(x) + g(x) = 0$.

Solution:

$f(x) = (x-r)(x-s) = x^2 - (r+s)x + rs$ and so $rs = -9$. Likewise $g(x) = (x+r)(x+s) = x^2 + (r+s)x + rs$, so $f(x) + g(x) = 2x^2 + 2rs = 2(x^2 - 9) = 0$ gives $x = \pm 3$.

Note: A lot of teams guessed that b and d were 9 and finished the solution from there. You need to show this, since there are a lot of possibilities for these two numbers.

2. Let $P(x)$ be a non-zero polynomial. Under what conditions on P do we have each of the following properties?

a. $P(P(x)) = 1$

b. $P(P(x)) = x$

c. $P(P(x)) = P(x)$

d. $P(P(x)) = (P(x))^2$

Solution :

a. A polynomial has form $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$. If we compose this function with itself its highest power will be $a_n(a_n x^n)^n = a_n^{n+1} x^{n^2}$. To be equal to one this must be zero, unless $n=0$. In this case $P(x) = c$, a constant. then $P(P(x)) = c$. So c must be 1. The only solution in $P(x) = 1$.

b. Again, looking at the highest power, we must have that $a_n(a_n x^n)^n = a_n^{n+1} x^{n^2} = x$ so $n=1$, and then $a_1^2=1$, so $a_1 = 1$ or $a_1 = -1$.

Case when $a_1 = 1$. Then $P(x) = x+c$, for some number c . Then $P(P(x)) = (x+c)+c = x+2c$, so $c=0$.

Case when $a_1 = -1$. then $P(x)=-x+c$ and $P(P(x)) = -(-x+c) + c = x$. So $P(x)=-x+c$ works for any x .

c. If the highest power of $P(x)$ is n , ie $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, then $P(P(x))$ has highest degree that looks like $a_n(a_n x^n)^n = a_n^{n+1} x^{n^2}$. We want this to be equal to $a_n x^n$. This says that $n^2 = n$, so that $n=0$ or $n=1$. The $n=0$ case is the constant function $P(x)=1$.

In the case when $n=1$, we get that $a(a x + b) + b = (a x + b)$ or $a^2 x + (ab + b) = ax + b$. For $a^2 = a$, we have $a = 0$ (back to the constant function), or $a=1$. Then from the constant terms we have $2b=b$, so $b=0$. So the only polynomials possible are $P(x) = x$ and $P(x) = C$, for any real number C .

d. In this case from the highest powers we have $a_n(a_n x^n)^n = a_n^{n+1} x^{n^2} =$
(Subscript[a, n] x^n)^2 = a_n^2 x^{2n}. This is only possible if $2n = n^2$, so $n = 0$ (constant function) or $n=2$.

If $P(x)=c$, then $P(P(x)) = c$ while $P(x)^2 = c^2$. So the only constant function is $c=1$ (or zero).

If $n=2$, then Subscript[a, 2]^3 = Subscript[a, 2]^2, so $a_2 = 1$. If $P(x) = x^2 + ax + b$, then $P(x)^2 = x^4 + 2ax^3 + (2b + a^2)x^2 + 2abx + b^2$.

Also $P(P(x)) = x^4 + 2ax^3 + (a + a^2 + b^2)x^2 + a^2x + b + ab + b^2$. Setting the coefficients equal we get the system of equations $b(a+1)=0$, $2ab = a^2$, and $2b = a + b^2$. From the first equation $b=0$, or $a=-1$. If $b=0$, then from the second equation $a=0$ as well, so $P(x) = x^2$. If $a=-1$, then the second equation gives $b=-1$, but this solution does not satisfy the last equation.

So $P(x)=1$ or $P(x) = x^2$

Note: On this problem many teams guessed some solutions. But in mathematics we also want to know if we have all the solutions. You should learn to ask this question and try to give an argument about why you have all the solutions,

3. You are given two sizes of ceramic tiles. There are 1×1 tiles, in white and red colors, and 1×3 tiles, in blue, green and orange colors. You can make patterns by stringing tiles together. For example you can make a 1×6 tile of red, orange, red, white tiles. Of course you could also tile a 1×6 tile using blue followed by green. You are also allowed to be boring and tile the 1×6 using all red tiles, if you wish. We also count using the same colors, but in a different order, as a new tiling.

How many different tilings are there of a $1 \times n$ tiles, for $n = 1, 2, 3, 4, 5$, and 6 ?

Solution:

Let $T(n)$ = number of tilings of a $1 \times n$ tiling.

For $n=1$, we can use one of the the white or red tiles. So 2 possible. $T(1)=2$.

For $n=2$ we have two choices for the leftmost tiles and 2 for the next tile, so $T(2) = 2 \times 2 = 4$.

For $n=3$, we can make the leftmost tiling a 1×1 tiling (2 choices), then we are left with a 1×2 tile to file, so $2 \times T(2) = 8$. Or we can put down a 1×3 tiling (3 choices) and be done. So $T(3) = 2 \times T(2) + 3 = 11$

Note: A lot of teams continued in this way, working up to 4,5 and 6. Below is a way to work out all of these at once using a function. Technically htis is called using recursion and is a commonly idea to simplify a procedure in both mathematics and computer programming.

Now for any larger tiling we can proceed as for $n=3$: If we start with a 1×1 tiling we have a $1 \times (n-1)$ area left to tile, so $2 \times T(n-1)$ ways to tile. If we start with a 1×3 we have, in the same way, $3 \times T(n-3)$ ways to tile. So $T(n)=2T(n-1)+3T(n-3)$. Using this we get:

$T(4)=2 \times 11+3 \times 2= 28$, and likewise $T(5)=68$ and $T(6)=169$.

4. The numbers from 1 to 1000 are written, in order, in a large circle. Starting at 1, every r -th number ($1, r+1, 2r+1$, etc) is crossed out. This is continued until a number is reached that has already been crossed out.

a. If $r=15$ what is the total number of numbers that have been crossed out?

b. In general, what is the total number of cross-outs?

Solution :

a. The crossed out numbers are $1, 16, 31, \dots, 991=1+ 15 \times 66$
 and second time around $6, 21, 36, \dots, 996 =6 + 15 \times 66$
 and the third time around $11, 26, 41, \dots, 986 = 11+15 \times 65$
 the next number would be $986+15=1000 = 1$, our first repeat.

There were $67+67+66 = 200$ numbers crossed out all together.

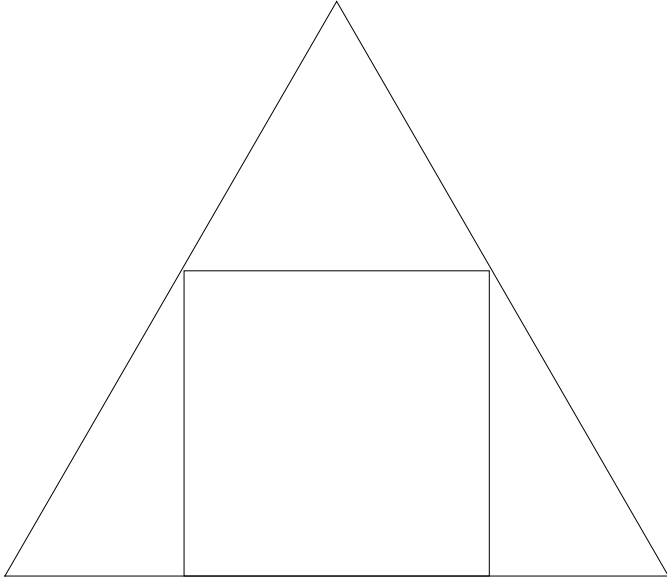
b. Suppose they match after n steps and on the n th step we are the same as an earlier m th step. Then we are asking when the numbers $(1+nr)-(1+mr)$ are divisible by 1000. This is equivalent to $(n-m) r$ being divisible by 1000.

i) If r and 1000 have no common divisors, then $n-m$ will be divisible by 1000, and so must be 1000. So, in this case, all the numbers will be crossed out.

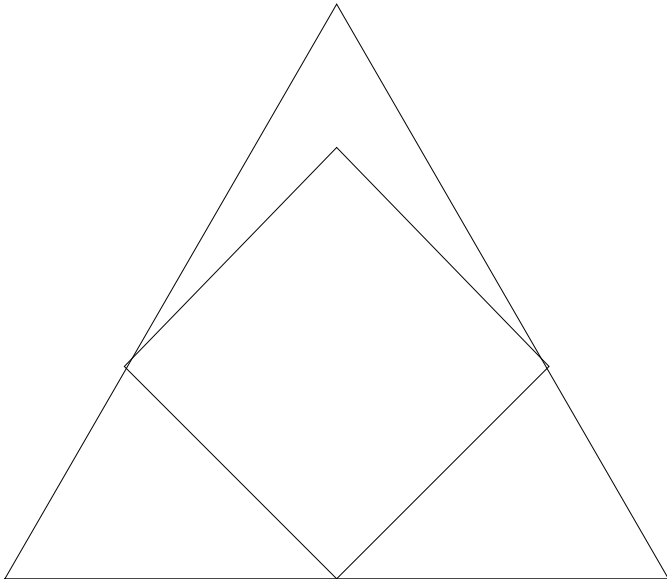
ii) Suppose $\text{GCD}(1000, r)$ is the greatest common divisor of r and 1000. Since $(n-m) r = (n-m) \text{GCD}(1000, r) p = 1000 k$, then we have that $n-m = 1000/\text{GCD}(1000, r)$. So the number crossed out is $1000/\text{GCD}(1000, r)$.

Note that in the previous case $r=15$ and $\text{GCD}(1000, 15)=5$. So the number crossed out is $1000/5 = 200$.

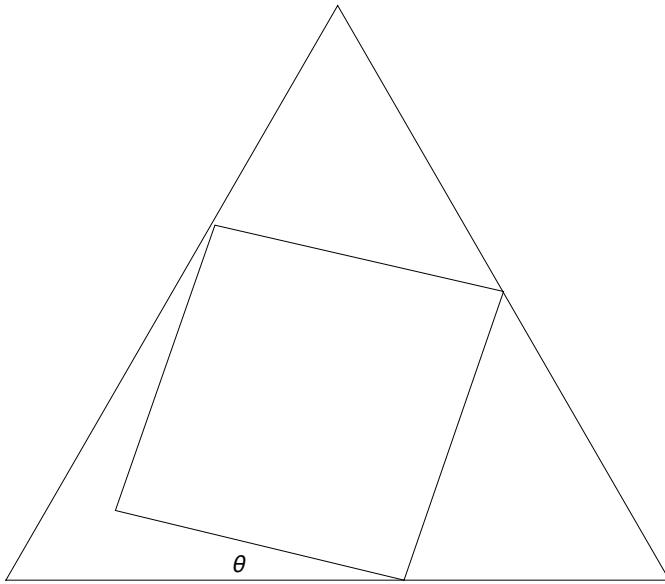
5. The largest square possible is placed in an equilateral triangle of side-length 2.
- a. If one side of the square sits on one side of the triangle, find the area of the largest such square.



- b. If the triangle is oriented as shown, then find the area of the largest such square.



- c. Finally, if the triangle is as shown, with $0 \leq \theta \leq 30$, find a formula for the area of the largest such triangle with square at the angle θ .



Solutions :

a.

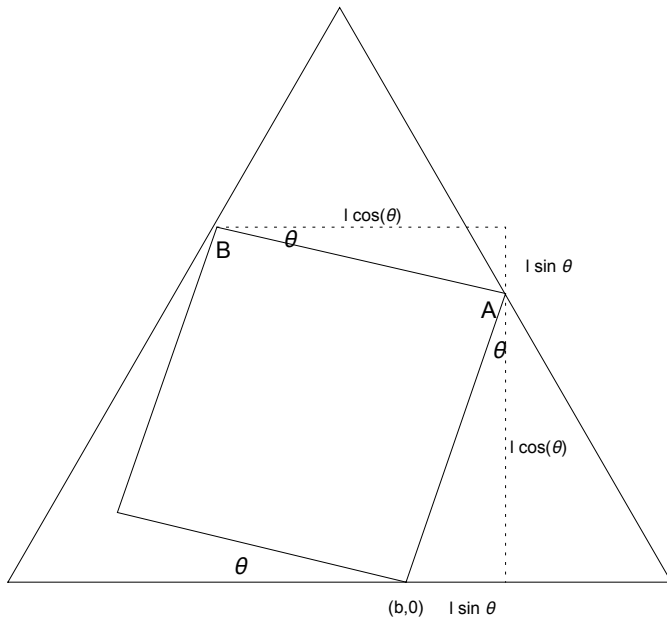
Let l be the side length. Place the triangle on a coordinate system with the base of the triangle sitting on the x -axis, with the origin at the bisection point of the base. Then the point where the top right hand corner of the square hits the triangle the coordinates are $(l/2, l)$. But the equation of the line thru the side of the triangle is $y = -\sqrt{3}(x - 1)$. substituting in the point $(l/2, l)$ gives $l = -\sqrt{3}(l/2 - 1)$ which reduces to $l = \frac{2\sqrt{3}}{2 + \sqrt{3}}$. So the area is $l^2 = \frac{12}{7 + 4\sqrt{3}}$.

b. (Using the same coordinate system as above:

In this case the line $y=x$ and the line $y = -\sqrt{3}(x - 1)$ intersect. Finding this intersection gives $x = \frac{\sqrt{3}}{1 + \sqrt{3}}$ for its x coordinate. Then $l^2 = x^2 + y^2, = 2x^2 = \frac{3}{2 + \sqrt{3}}$

Note : A lot of teams used trigonometry on this one. As you can see, that was not necessary on this problem.

c. We continue using the same coordinate system as above, and get the following points of intersection. We start by assuming the point where the square hits the base of the triangle has coordinates $(b, 0)$.



Then the point labeled A must have coordinates $(b+l \sin \theta, l \cos \theta)$ and the point labeled B is $(b+l \sin \theta - l \cos \theta, l \cos \theta + l \sin \theta)$.

Since A is on the line $y = -\sqrt{3}(x - 1)$, we can plug in the coordinates $(b+l \sin \theta, l \cos \theta)$ and simplify to $l(\cos \theta + \sqrt{3} \sin \theta) = \sqrt{3} - \sqrt{3} b$.

Likewise, since B is on the line $y = \sqrt{3}x + \sqrt{3}$, we can get that $l((1 + \sqrt{3}) \cos \theta + (1 - \sqrt{3}) \sin \theta) = \sqrt{3} + \sqrt{3} b$

Adding these two equations results in $l(\cos \theta(2 + \sqrt{3}) + \sin \theta) = 2\sqrt{3}$ or, $l = \frac{2\sqrt{3}}{(2 + \sqrt{3}) \cos \theta + \sin \theta}$. Then

the area is $(2 \text{ Sqrt}[3]) / (2 + \text{Sqrt}[3]) \cos \theta + \sin \theta)^2$.

Checking against the previous computations: (Using Mathematica)

$$f[\theta_] := \frac{2 \sqrt{3}}{(2 + \sqrt{3}) \text{Cos}[\theta] + \text{Sin}[\theta]}$$

Checking part a ($\theta = 0$)

$$f[0]^2 == \frac{12}{7 + 4 \sqrt{3}}$$

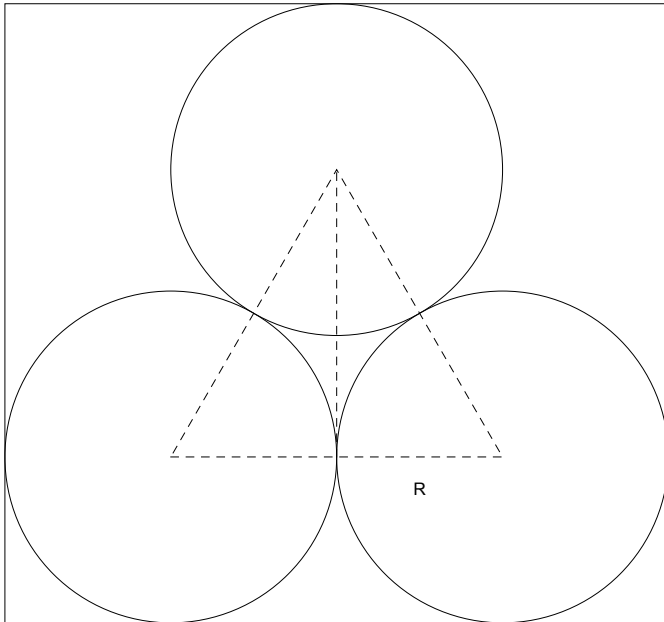
True

Checking part b ($\theta = 15$ degrees = $\pi/12$ radians, where θ is measured off the left side rather than the base.)

$$f[\pi/12.]^2 == \frac{3.}{2 + \sqrt{3}}$$

True

6. In the figure shown the circles are tangent to one-another and to the sides of the rectangle, as shown. If each of the circles has the same radius, R ,
- what is the area of the enclosing rectangle?
 - What is the area of the region trapped between the three circles?



- Connecting the centers of the circles we obtain an equilateral triangle of side length $2R$, as shown. The right half of this triangle is a 30-60-90 triangle and so the altitude is $\sqrt{3} R$. Then the width of the rectangle is $4R$ and its height is $2R + \sqrt{3} R$ for an area of $2(2 + \sqrt{3}) R^2$
- The area of the triangle is $1/2 (2R)(\sqrt{3} R) = \sqrt{3} R^2$. From this we need to subtract 3 sectors of the circles with center angle $\frac{\pi}{3}$. So the area is $\sqrt{3} R^2 - 3(\pi R^2 (1/6)) = R^2(\sqrt{3} - \frac{\pi}{2})$